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Dilatonic quantum multi-brane-worlds.

Shin'ichi NOJIRI[1], Sergei D. ODINTSOV[♠,♣2],
and K.E. OSETRIN[♠3]

Department of Applied Physics
National Defence Academy, Hashirimizu Yokosuka 239, JAPAN

♠ *Tomsk State Pedagogical University, 634041 Tomsk,RUSSIA*

♣ *Instituto de Fisica de la Universidad de Guanajuato*
Apdo.Postal E-143, 37150 Leon, Gto., MEXICO

ABSTRACT

d5 dilatonic gravity action with surface counterterms motivated by AdS/CFT correspondence and with contributions of brane quantum CFTs is considered around AdS-like bulk. The effective equations of motion are constructed. They admit two (outer and inner) or multi-brane solutions where brane CFTs may be different. The role of quantum brane CFT is in inducing of complicated brane dilatonic gravity. For exponential bulk potentials the number of AdS-like bulk spaces is found in analytical form. The correspondent flat or curved (de Sitter or hyperbolic) dilatonic two branes are created, as a rule, thanks to quantum effects. The observable early Universe may correspond to inflationary brane. The found dilatonic quantum two brane-worlds usually contain the naked singularity but in couple explicit examples the curvature is finite and horizon (corresponding to wormhole-like space) appears.

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[1]email: nojiri@cc.nda.ac.jp
[2]email: odintsov@ifug5.ugto.mx, odintsov@itp.uni-leipzig.de
[3]e-mail: osetrin@tspu.edu.ru

1 Introduction

Recent booming activity in the study of brane-worlds is caused by several reasons. First, gravity on 4d brane embedded in higher dimensional AdS-like Universe may be localized [1, 2]. Second, the way to resolve the mass hierarchy problem appears[1]. Third, the new ideas on cosmological constant problem solution come to game [7, 8]. Very incomplete list of references[3, 4] (and references therein) mainly on the cosmological aspects of brane-worlds is growing every day.

The essential element of brane-world models is the presence in the theory of two free parameters (bulk cosmological constant and brane tension, or brane cosmological constant). The role of brane cosmological constant is to fix the position of the brane in terms of tension (that is why brane cosmological constant and brane tension are almost the same thing). Being completely consistent and mathematically reasonable, such way of doing things may look not completely satisfactory. Indeed, the physical origin (and prediction) of brane tension in terms of some dynamical mechanism may be required.

The ideology may be different, in the spirit of refs.[6, 5]. One considers the addition of surface counterterms to the original action on AdS-like space. These terms are responcible for making the variational procedure to be well-defined (in Gibbons-Hawking spirit) and for elimination of the leading divergences of the action. Brane tension is not considered as free parameter anymore but it is fixed by the condition of finiteness of spacetime when brane goes to infinity. Of course, leaving the theory in such form would rule out the possibility of consistent brane-world solutions existance. Fortunately, other parameters contribute to brane tension. If one considers that there is quantum CFT living on the brane (which is more close to the spirit of AdS/CFT correspondence[9]) then such CFT produces conformal anomaly (or anomaly induced effective action). This contributes to brane tension. As a result dynamical mechanism to get brane-world with flat or curved (de Sitter or Anti-de Sitter) brane appears. The curvature of such 4d Universe is expressed in terms of some dimensional parameter l which usually appears in AdS/CFT set-up and of content of quantum brane matter. In other words, brane-world is the consequence of the fact (verified experimentally by everybody life) of the presence of matter on the brane! For example, sign of conformal anomaly terms for usual matter is such that in one-brane case the de Sitter (ever expanding, inflationary) Universe is preferrable solution of brane equation[4].

The scenario of refs.[6, 5] may be extended to the presence of dilaton(s) as it was done in ref.[10] or to formulation of quantum cosmology in Wheeler-De Witt form [11]. Then whole scenario looks even more related with AdS/CFT correspondence as dilatonic gravity naturally follows as bosonic sector of d5 gauged supergravity. Moreover, the extra prize-in form of dynamical determination of 4d boundary value of dilaton-appears. In ref.[10] the quantum dilatonic one brane Universe has been presented with possibility to get inflationary or hyperbolic or flat brane with dynamical determination of brane dilaton. The interesting question is related with generalization of such scenario in dilatonic gravity for multi-brane case. This will be the purpose of present work.

In the next section we present general action of d5 dilatonic gravity with surface counterterms and quantum brane CFT contribution. This action is convenient for description of brane-worlds where bulk is AdS-like spacetime. There could be one or two (flat or curved) branes in the theory. As it was already mentioned the brane tension is fixed in our approach, instead of it the effective brane tension is induced by quantum effects. Section three is devoted to formulation of effective bulk-brane field equations. The explicit analytical solution of bulk equation for number of exponential bulk potentials is presented. The lengthy analysis of 4d brane equations shows the possibility to have two (inner and outer) branes associated with each of above bulk solutions. It is interesting that quantum created branes can be flat, or de Sitter (inflationary) or hyperbolic. The role of quantum brane matter corrections in getting of such branes is extremely important. Nevertheless, there are few particular cases where such branes appear on classical level, i.e. without quantum corrections. In section four we briefly describe how to get generalization of above solutions for quantum dilatonic multi-brane-worlds with more than two branes. Brief summary of results is given in final section where also the study of character of singularities for proposed two-brane solutions is presented. In most cases, as usually occurs in AdS dilatonic gravity, the solutions contain the naked singularity. However, in couple cases the scalar curvature is finite and there is horizon. The corresponding 4d branes may be interpreted as wormhole.

[4]Similar mechanism for anomaly driven inflation in usual 4d world has been invented by Starobinsky[15] and generalized for dilaton presence in refs.[17]

2 Dilatonic gravity action with brane quantum corrections

Let us present the initial action for dilatonic AdS gravity under consideration. The metric of (Euclidean) AdS has the following form:

$$ds^2 = dz^2 + \mathrm{e}^{2\tilde{A}(z)} \sum_{i,j=1}^{4} \hat{g}_{ij} dx^i dx^j \ . \tag{1}$$

Here $\hat{g}_{ij}$ is the metric of the Einstein manifold, which is defined by $r_{ij} = k\hat{g}_{ij}$, where r_{ij} is the Ricci tensor constructed with $\hat{g}_{ij}$ and k is a constant. One can consider two copies of the regions given by $z < z_0$ and glue two regions putting a brane at $z = z_0$. More generally, one can consider two copies of regions $\tilde{z}_0 < z < z_0$ and glue the regions putting two branes at $z = \tilde{z}_0$ and $z = z_0$. Hereafter we call the brane at $z = \tilde{z}_0$ as "inner' brane and that at $z = z_0$ as "outer" brane.

Let us first consider the case with only one brane at $z = z_0$ and start with Euclidean signature action S which is the sum of the Einstein-Hilbert action $S_{\rm EH}$ with kinetic term and potential $V(\phi) = \frac{12}{l^2} + \Phi(\phi)$ for dilaton ϕ, the Gibbons-Hawking surface term $S_{\rm GH}$, the surface counter term S_1 and the trace anomaly induced action W[5]:

$$\begin{aligned}
S &= S_{\rm EH} + S_{\rm GH} + 2S_1 + W, && (2)\\
S_{\rm EH} &= \frac{1}{16\pi G}\int d^5x\sqrt{g_{(5)}}\left(R_{(5)} - \frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi + \frac{12}{l^2} + \Phi(\phi)\right), && (3)\\
S_{\rm GH} &= \frac{1}{8\pi G}\int d^4x\sqrt{g_{(4)}}\nabla_\mu n^\mu, && (4)\\
S_1 &= -\frac{1}{16\pi Gl}\int d^4x\sqrt{g_{(4)}}\left(\frac{6}{l} + \frac{l}{4}\Phi(\phi)\right), && (5)\\
W &= b\int d^4x\sqrt{\widetilde{g}}\widetilde{F}A \\
&\quad + b'\int d^4x\sqrt{\widetilde{g}}\left\{A\left[2\,\widetilde{\Box}^2 + \widetilde{R}_{\mu\nu}\widetilde{\nabla}_\mu\widetilde{\nabla}_\nu - \frac{4}{3}\widetilde{R}\,\widetilde{\Box}^2 + \frac{2}{3}(\widetilde{\nabla}^\mu\widetilde{R})\widetilde{\nabla}_\mu\right]A\right.
\end{aligned}$$

[5]For the introduction to anomaly induced effective action in curved space-time (with torsion), see section 5.5 in [12]. This anomaly induced action is due to brane CFT living on the boundary of dilatonic AdS-like space.

$$+\left(\tilde{G}-\frac{2}{3}\tilde{\Box}\tilde{R}\right)A\Big\}$$

$$-\frac{1}{12}\left\{b''+\frac{2}{3}(b+b')\right\}\int d^4x\sqrt{\tilde{g}}\left[\tilde{R}-6\,\tilde{\Box}A-6(\tilde{\nabla}_\mu A)(\tilde{\nabla}^\mu A)\right]^2$$

$$+C\int d^4x\sqrt{\tilde{g}}A\phi\left[\tilde{\Box}^2+2\tilde{R}_{\mu\nu}\tilde{\nabla}_\mu\tilde{\nabla}_\nu-\frac{2}{3}\tilde{R}\,\tilde{\Box}^2+\frac{1}{3}(\tilde{\nabla}^\mu\tilde{R})\tilde{\nabla}_\mu\right]\phi \quad (6)$$

Here the quantities in the 5 dimensional bulk spacetime are specified by the suffices $_{(5)}$ and those in the boundary 4 dimensional spacetime are specified by $_{(4)}$. The factor 2 in front of S_1 in (2) is coming from that we have two bulk regions which are connected with each other by the brane. In (4), n^μ is the unit vector normal to the boundary. In (6), one chooses the 4 dimensional boundary metric as

$$g_{(4)\mu\nu}=\mathrm{e}^{2A}\tilde{g}_{\mu\nu}, \quad (7)$$

and we specify the quantities given by $\tilde{g}_{\mu\nu}$ by using $\tilde{}$. G ($\tilde{G}$) and F ($\tilde{F}$) are the Gauss-Bonnet invariant and the square of the Weyl tensor, which are given as

$$\begin{aligned} G &= R^2-4R_{ij}R^{ij}+R_{ijkl}R^{ijkl}, \\ F &= \frac{1}{3}R^2-2R_{ij}R^{ij}+R_{ijkl}R^{ijkl}\ , \end{aligned} \quad (8)$$

[6] In the effective action (6) induced by brane quantum matter, with N scalar, $N_{1/2}$ spinor, N_1 vector fields, N_2 ($=0$ or 1) gravitons and N_{HD} higher derivative conformal scalars, b, b' and b'' are

$$b = \frac{N+6N_{1/2}+12N_1+611N_2-8N_{\mathrm{HD}}}{120(4\pi)^2}$$

---

[6]We use the following curvature conventions:

$$\begin{aligned} R &= g^{\mu\nu}R_{\mu\nu} \\ R_{\mu\nu} &= R^\lambda{}_{\mu\lambda\nu} \\ R^\lambda{}_{\mu\rho\nu} &= -\Gamma^\lambda_{\mu\rho,\nu}+\Gamma^\lambda_{\mu\nu,\rho}-\Gamma^\eta_{\mu\rho}\Gamma^\lambda_{\nu\eta}+\Gamma^\eta_{\mu\nu}\Gamma^\lambda_{\rho\eta} \\ \Gamma^\eta_{\mu\lambda} &= \frac{1}{2}g^{\eta\nu}\left(g_{\mu\nu,\lambda}+g_{\lambda\nu,\mu}-g_{\mu\lambda,\nu}\right)\ . \end{aligned}$$

$$
\begin{aligned}
b' &= -\frac{N + 11N_{1/2} + 62N_1 + 1411N_2 - 28N_{\rm HD}}{360(4\pi)^2} \ , \\
b'' &= 0 \ .
\end{aligned} \tag{9}
$$

As usually, b'' may be changed by the finite renormalization of local counterterm in gravitational effective action. As it was the case in ref.[10], the term proportional to $\left\{b'' + \frac{2}{3}(b + b')\right\}$ in (6), and therefore b'', does not contribute to the equations of motion. Note that CFT matter induced effective action may be considered as brane dilatonic gravity.

For typical examples motivated by AdS/CFT correspondence[9] one has:

a) $\mathcal{N} = 4$ $SU(N)$ SYM theory

$$
b = -b' = \frac{C}{4} = \frac{N^2 - 1}{4(4\pi)^2} \ , \tag{10}
$$

b) $\mathcal{N} = 2$ $Sp(N)$ theory

$$
b = \frac{12N^2 + 18N - 2}{24(4\pi)^2} \ , \quad b' = -\frac{12N^2 + 12N - 1}{24(4\pi)^2} \ . \tag{11}
$$

One can write the corresponding expression for dilaton coupled spinor matter [14] which also has non-trivial (slightly different in form) dilatonic contribution to CA than in case of holografic conformal anomaly[13] for $\mathcal{N} = 4$ super Yang-Mills theory.

Let us consider the case where there are two branes at $z = \tilde z_0$ and $z = z_0$, adding the action corresponding to the brane at $z = \tilde z_0$ to the action in (2):

$$
\begin{aligned}
S_{\rm two\ branes} &= S + \tilde S_{\rm GH} + 2\tilde S_1 + \tilde W, && (12) \\
\tilde S_{\rm GH} &= \frac{1}{8\pi G}\int d^4x\sqrt{g_{(4)}}\nabla_\mu n^\mu, && (13) \\
\tilde S_1 &= \frac{1}{16\pi G l}\int d^4x\sqrt{g_{(4)}}\left(\frac{6}{l} + \frac{l}{4}\Phi(\phi)\right), && (14) \\
\tilde W &= \tilde b\int d^4x\sqrt{\tilde g}\tilde F A && (15) \\
& +\tilde b'\int d^4x\sqrt{\tilde g}\left\{A\left[2\,\widetilde{\Box}^2 + \tilde R_{\mu\nu}\widetilde\nabla_\mu\widetilde\nabla_\nu - \frac{4}{3}\tilde R\,\widetilde{\Box}^2 + \frac{2}{3}(\widetilde\nabla^\mu\tilde R)\widetilde\nabla_\mu\right]A \right. \\
& \left. + \left(\tilde G - \frac{2}{3}\,\widetilde{\Box}\tilde R\right)A\right\}
\end{aligned}
$$

$$-\frac{1}{12}\left\{\tilde{b}''+\frac{2}{3}(\tilde{b}+\tilde{b}')\right\}\int d^4x\sqrt{\tilde{g}}\left[\tilde{R}-6\,\tilde{\Box}A-6(\widetilde{\nabla}_\mu A)(\widetilde{\nabla}^\mu A)\right]^2$$
$$+\tilde{C}\int d^4x\sqrt{\tilde{g}}A\phi\left[\tilde{\Box}^2+2\tilde{R}_{\mu\nu}\widetilde{\nabla}_\mu\widetilde{\nabla}_\nu-\frac{2}{3}\tilde{R}\,\tilde{\Box}^2+\frac{1}{3}(\widetilde{\nabla}^\mu\tilde{R})\widetilde{\nabla}_\mu\right]\phi\,.$$

We should note that the relative sign of $\tilde{S}_1$ is different from S_1. The parameters $\tilde{b}$, $\tilde{b}'$, $\tilde{b}''$ and $\tilde{C}$ correspond to the matter which may be different from the outer brane one on the inner brane as in (9). Hence, the situation with different CFTs on the branes may be considered. Having the action at hands one can study its dynamics.

3 Dilatonic quantum brane-worlds

Let us start the consideration of field equations for two-branes model. First of all, one defines a new coordinate z by

$$z=\int dy\sqrt{f(y)}, \tag{16}$$

and solves y with respect to z. Then the warp factor is $\mathrm{e}^{2\hat{A}(z,k)}=y(z)$. Here one assumes the metric of 5 dimensional space time as follows:

$$ds^2=f(y)dy^2+y\sum_{i,j=1}^{4}\hat{g}_{ij}(x^k)dx^i dx^j. \tag{17}$$

Assuming that ϕ depends only on y: $\phi=\phi(y)$, we obtain the following equations of motion in the bulk:

$$\begin{aligned} 0 &= \frac{3}{2y^2}-\frac{2kf}{y}-\frac{1}{4}\left(\frac{d\phi}{dy}\right)^2-\left(\frac{6}{l^2}+\frac{1}{2}\Phi(\phi)\right)f, && (18)\\ 0 &= \frac{d}{dy}\left(\frac{y^2}{\sqrt{f}}\frac{d\phi}{dy}\right)+\Phi'(\phi)y^2\sqrt{f}\,. && (19)\end{aligned}$$

On the other hand, on the (outer) brane, we obtain the following equations:

$$0 = \frac{48l^4}{16\pi G}\left(\partial_z A-\frac{1}{l}-\frac{l}{24}\Phi(\phi)\right)\mathrm{e}^{4A}+b'\left(4\partial_\sigma^4 A-16\partial_\sigma^2 A\right)$$

$$
\begin{aligned}
& -4(b+b')\left(\partial_\sigma^4 A + 2\partial_\sigma^2 A - 6(\partial_\sigma A)^2\partial_\sigma^2 A\right) \\
& +2C\left(\partial_\sigma^4\phi - 4\partial_\sigma^2\phi\right) , \qquad (20) \\
0 \;=\; & -\frac{l^4}{8\pi G}\mathrm{e}^{4A}\partial_z\phi - \frac{3l^3}{4\pi G}\mathrm{e}^{4A}\Phi'(\phi) \\
& +C\left\{A\left(\partial_\sigma^4\phi - 4\partial_\sigma^2\phi\right) + \partial_\sigma^4(A\phi) - 4\partial_\sigma^2(A\phi)\right\} \ . \qquad (21)
\end{aligned}
$$

For inner brane, one gets

$$
\begin{aligned}
0 \;=\; & -\frac{48l^4}{16\pi G}\left(\partial_z A - \frac{1}{l} - \frac{l}{24}\Phi(\phi)\right)\mathrm{e}^{4A} + \tilde{b}'\left(4\partial_\sigma^4 A - 16\partial_\sigma^2 A\right) \\
& -4(\tilde{b}+\tilde{b}')\left(\partial_\sigma^4 A + 2\partial_\sigma^2 A - 6(\partial_\sigma A)^2\partial_\sigma^2 A\right) \\
& +2\tilde{C}\left(\partial_\sigma^4\phi - 4\partial_\sigma^2\phi\right) , \qquad (22) \\
0 \;=\; & \frac{l^4}{8\pi G}\mathrm{e}^{4A}\partial_z\phi + \frac{3l^3}{4\pi G}\mathrm{e}^{4A}\Phi'(\phi) \\
& +\tilde{C}\left\{A\left(\partial_\sigma^4\phi - 4\partial_\sigma^2\phi\right) + \partial_\sigma^4(A\phi) - 4\partial_\sigma^2(A\phi)\right\} \ . \qquad (23)
\end{aligned}
$$

In (20) and (21), using the change of the coordinate: $dz = \sqrt{f}dy$ and choosing $l^2\mathrm{e}^{2\hat{A}(z,k)} = y(z)$ one uses the form of the metric as

$$
ds^2 = dz^2 + \mathrm{e}^{2A(z,\sigma)}\tilde{g}_{\mu\nu}dx^\mu dx^\nu \ , \quad \tilde{g}_{\mu\nu}dx^\mu dx^\nu \equiv l^2\left(d\sigma^2 + d\Omega_3^2\right) \ . \qquad (24)
$$

Here $d\Omega_3^2$ corresponds to the metric of 3 dimensional unit sphere. Then for the unit sphere ($k=3$)

$$
A(z,\sigma) = \hat{A}(z,k=3) - \ln\cosh\sigma \ , \qquad (25)
$$

for the flat Euclidean space ($k=0$)

$$
A(z,\sigma) = \hat{A}(z,k=0) + \sigma \ , \qquad (26)
$$

and for the unit hyperboloid ($k=-3$)

$$
A(z,\sigma) = \hat{A}(z,k=-3) - \ln\sinh\sigma \ . \qquad (27)
$$

We now identify A and $\tilde{g}$ in (24) with those in (7). Then we find $\tilde{F} = \tilde{G} = 0$, $\tilde{R} = \frac{6}{l^2}$ etc.

Using (18) and (19), one can delete f from the equations and can obtain an equation that contains only the dilaton field ϕ (and ,of course, bulk potential):

$$\begin{aligned} 0 &= \left\{\frac{5k}{2} - \frac{k}{4}y^2\left(\frac{d\phi}{dy}\right)^2 + \left(\frac{3}{2}y - \frac{y^3}{6}\left(\frac{d\phi}{dy}\right)^2\right)\left(\frac{6}{l^2} + \frac{1}{2}\Phi(\phi)\right)\right\}\frac{d\phi}{dy} \\ &\quad + \frac{y^2}{2}\left(\frac{2k}{y} + \frac{6}{l^2} + \frac{1}{2}\Phi(\phi)\right)\frac{d^2\phi}{dy^2} + \left(\frac{3}{4} - \frac{y^2}{8}\left(\frac{d\phi}{dy}\right)^2\right)\Phi'(\phi) \ . \end{aligned} \tag{28}$$

Our choice for dilaton and bulk potential admitting the analytical solution is

$$\phi(y) = p_1 \ln(p_2 y) \tag{29}$$

$$\Phi(\phi) = -\frac{12}{l^2} + c_1 \exp(a\phi) + c_2 \exp(2a\phi) \ , \tag{30}$$

where a, p_1, p_2, c_1, c_2 are some constants. When $p_1 = \pm\frac{1}{\sqrt{6}}$, Eq.(28) is always satisfied but from Eq.(19), we find that $f(y)$ identically vanishes. Therefore we should assume $p_1 \neq \pm\frac{1}{\sqrt{6}}$. Then we find the following set of exact bulk solutions

case 1 $\quad c_1 = \frac{6kp_2p_1^2}{3-2p_1^2} \ , \quad c_2 = 0 \ , \quad a = -\frac{1}{p_1} \ , \quad p_1 \neq \pm\sqrt{6}$

$$f(y) = \frac{3-2p_1^2}{4ky} \tag{31}$$

case 2 $\quad c_1 = -6kp_2 \ , \quad a = \pm\frac{1}{\sqrt{3}} \ , \quad p_1 = \mp\sqrt{3}$

$$f(y) = \frac{3}{\frac{2c_2}{p_2^2} - 4ky} \tag{32}$$

case 3 $\quad c_2 = 3kp_2 \ , \quad a = \pm\frac{1}{\sqrt{3}} \ , \quad p_1 = \mp\frac{\sqrt{3}}{2}$

$$f(y) = \frac{21\sqrt{p_2}}{8\sqrt{y}\left(c_1 y + 7k\sqrt{p_2 y}\right)} \ . \tag{33}$$

In the coordinate system in (17), Eq.(21) for outer brane has the following form:

$$0 = -\frac{y_0^2}{8\pi G\sqrt{f(y_0)}}\partial_y\phi - \frac{3y_0^2}{4\pi G l}\Phi'(\phi) + 6C\phi_0 \ , \tag{34}$$

and (22) for innner brane

$$0=\frac{\tilde{y}_0^2}{8\pi G\sqrt{f(\tilde{y}_0)}}\partial_y\phi+\frac{3\tilde{y}_0^2}{4\pi Gl}\Phi'(\tilde{\phi}_0)+6\tilde{C}\tilde{\phi}_0\ . \tag{35}$$

Here ϕ_0 ($\tilde{\phi}_0$) is the value of the dilaton ϕ on the outer (inner) brane. We also find Eq.(20) for outer brane has the following form:

$$0=\frac{3y_0^2}{16\pi G}\left(\frac{1}{2y_0\sqrt{f(y_0)}}-\frac{1}{l}-\frac{l}{24}\Phi(\phi)\right)+8b' \tag{36}$$

for $k\neq 0$ case and

$$0=\frac{3y_0^2}{16\pi G}\left(\frac{1}{2y_0\sqrt{f(y_0)}}-\frac{1}{l}-\frac{l}{24}\Phi(\phi)\right) \tag{37}$$

for $k=0$ case. For the inner brane (22) for $k\neq 0$ has the form of

$$0=-\frac{3\tilde{y}_0^2}{16\pi G}\left(\frac{1}{2\tilde{y}_0\sqrt{f(\tilde{y}_0)}}+\frac{1}{l}+\frac{l}{24}\Phi(\tilde{\phi}_0)\right)+8\tilde{b}'\ . \tag{38}$$

The equation for $k=0$ case is identical with that of the outer brane in (37).

Case 1 solution.

First we consider case 1 in (31). Since $f(y)$ should be positive (we should also note $y>0$), one gets

$$q^2\equiv\frac{4k}{3-2p_1^2}>0\ ,\quad (q>0)\ . \tag{39}$$

In (39), we can also consider the limit of $k\to 0$ by keeping q finite, i.e., $p_1^2\to\frac{3}{2}$.

When $k\neq 0$, Eqs.(36) and (34) have the following form:

$$\begin{aligned}-8b' &= F_1(y_0)\equiv\frac{3}{16\pi G}\left(\frac{q}{2}y_0^{\frac{3}{2}}-\frac{1}{2l}y_0^2-\frac{q^2p_1^2ly_0}{16}\right)\\ &= -\frac{3}{16\pi G}\frac{y_0}{2l}\left(y_0^{\frac{1}{2}}-\frac{1+\sqrt{1-\frac{p_1^2l^2}{2}}}{2}q\right)\left(y_0^{\frac{1}{2}}-\frac{1-\sqrt{1-\frac{p_1^2l^2}{2}}}{2}q\right) \quad (40)\\ 0 &= -\frac{q}{8\pi G}y_0^{\frac{5}{2}}+\frac{9lp_1q^2}{8\pi G}y_0+6C\phi_0\ . \quad (41)\end{aligned}$$

and Eqs.(22) and (23) are

$$8\tilde{b}' = F_1(y_0) \equiv \frac{3}{16\pi G}\left(\frac{q}{2}y_0^{\frac{3}{2}} - \frac{1}{2l}y_0^2 - \frac{q^2p_1^2 l y_0}{16}\right) \quad (42)$$

$$0 = \frac{p_1 q}{8\pi G}y_0^{\frac{3}{2}} - \frac{9lp_1q^2}{8\pi G}y_0 + 6\tilde{C}\tilde{\phi}_0 \ . \quad (43)$$

Since p_2 is absorbed into the definition of q in (40) and (42), Eqs.(41) and (43) can be regarded as the equation which determines p_2 or $\frac{\phi_0}{p_1}\ln(p_2 y_0)$ and $\frac{\tilde{\phi}_0}{p_1} = \ln(p_2\tilde{y}_0)$. We now investigate the properties of $F_1(y_0)$ as a function of y_0. The asymptotic behaviors are given by

$$F_1(y_0) \to -\frac{3}{16\pi G}\cdot\frac{p_1^2q^2l}{16}y_0 < 0 \qquad \text{when } y_0 \to +0 \quad (44)$$

$$F_1(y_0) \to -\frac{3}{16\pi G}\cdot\frac{1}{2l}y_0^2 < 0 \qquad \text{when } y_0 \to +\infty \ . \quad (45)$$

Since

$$F_1'(y_0) = \frac{3}{16\pi G}\left(\frac{3q}{4}y^{\frac{1}{2}} - \frac{1}{l}y_0 - \frac{p_1^2q^2l}{16}\right) \ , \quad (46)$$

$F_1(y_0)$ has extrema when

$$0 = y_0 - \frac{3ql}{4}y_0^{\frac{1}{2}} + \frac{p_1^2q^2l^2}{16} \ , \quad (47)$$

whose solutions are given by

$$y_0^{\frac{1}{2}} = y_\pm^{\frac{1}{2}} \equiv \frac{3ql}{8}\left(1 \pm \sqrt{1 - \frac{4p_1^2}{9}}\right) \ . \quad (48)$$

Therefore if

$$|p_1| > \frac{3}{2} \ , \quad (49)$$

Eq.(47) does not have any solution and $F_1(y_0)$ is monotonically decreasing function of y_0. Then Eqs.(44) and (45) tell that there is no solution of the brane equation (40) for negative b' in case of (49). On the other hand, when

$$|p_1| < \frac{3}{2} \ , \quad (50)$$

substituting (48) into the expression for $F_1(y_0)$ in (40), one gets

$$F_1(y_\pm) = \frac{3}{16\pi G}\frac{3^4 q^4 l^3}{2\cdot 8^4}\left(\sqrt{1-\frac{4p_1^2}{9}}\pm 1\right)\left(\sqrt{1-\frac{4p_1^2}{9}}\mp \frac{1}{3}\right) . \tag{51}$$

Then we find $y_0 = y_+$ corresponds to the maximum of $F_1(y_0)$. The maximum is positive $F_1(y_+) > 0$ if $\sqrt{1-\frac{4p_1^2}{9}} - \frac{1}{3} > 0$, that is,

$$p_1^2 < 2 . \tag{52}$$

In case of (52), if

$$F_1(y_+) \geq -8b' , \tag{53}$$

Eq.(40) has a solution, that is, there can be a brane. We can also consider an inner brane which lies at $y = y_1 < y_0$. For the inner brane, the relative sign of $\tilde{b}'$ and b' is changed in the equation corresponding to (40). Then if

$$F_1(y_-) \leq 8\tilde{b}' , \tag{54}$$

there can be an inner brane. Then if both of (53) and (54) hold, we can have two brane dilatonic solution. In case of two brane solution, there might be, in general, a problem in the consistency between (41) and (43). If we impose both of (41) and (43), they can be regarded as the equations which determine p_1 and p_2 (we should note that p_2 is implicitly contained in ϕ_0 and $\tilde{\phi}_0$). In the classical limit, where $C = 0$, there disappear the terms containing p_2 (or ϕ_0 and $\tilde{\phi}_0$). Then it seems to be non-trivial if there exists any solution which satisfies both of (53) and (54).

We now consider the classical limit in $k \neq 0$ case, where $b' = C = \tilde{b}' = \tilde{C} = 0$ and (40) and (42) become identical. Then the solutions of Eqs.(40) and (42) are given by

$$y_0^{\frac{1}{2}} = \left(1 \pm \sqrt{1-\frac{p_1^2}{2}}\right)\frac{ql}{2} . \tag{55}$$

Since both of the solutions are positive, we can regard smaller one ($-$ sign in (55)) as expressing inner brane and larger one (+ sign) as outer brane. On the other hand, Eqs.(41) and (43) have the following form:

$$0 = \frac{p_1 q^2 l y_0}{2}\left(17 \mp \sqrt{1-\frac{p_1^2}{2}}\right) . \tag{56}$$

In (56), the upper sign $(-)$ corresponds to the outer brane and the lower one $(+)$ to the inner brane. We should note that there is no solution, except trivial one, for inner brane. This would tell that we need the quantum correction from brane matter in order to obtain the two brane dilatonic inflationary Universe where observable world may be associated with one of inflationary branes.

When $k=0$ in case 1, as discussed before, if q is finite

$$p_1^2 \to \frac{3}{2} \ . \tag{57}$$

Then Eq.(37) can be rewritten in the following form:

$$0 = y_0 - q l y_0^{\frac{1}{2}} + \frac{3q^2 l^2}{16} \ , \tag{58}$$

which has two solutions:

$$y_0^{\frac{1}{2}} = \frac{3ql}{4} \ , \quad \frac{ql}{4} \ . \tag{59}$$

These two solutions might be regarded as two brane solutions. On the other hand, the form of Eq.(34) for $k=0$ case is identical with that of $k \neq 0$ in (41), which can be solved with respect to ϕ_0 or p_2 in one brane solution. However, in case $k=0$, the value of p_1 is fixed by (57). In the classical limit of $k=0$ case, Eq.(37) can be rewritten in the form of (58) and there appear solutions in (59). Eq.(34) is, however, not satisfied. Eq.(34) has a form of (56) but Eq.(57) does not satisfy (56). This demonstrates the role of quantum effects in realization of dilatonic inflationary two brane-world Universe.

Case 2 solution.

We now consider the case 2 in (32). Defining $\tilde{c}_2$ as

$$\tilde{c}_2 \equiv \frac{c_2}{p_2^2} \ , \tag{60}$$

Eqs.(36) and (34) for the outer brane have the following form (when $k \neq 0$):

$$\begin{aligned} -8b' &= F_2(y_0) \equiv \frac{3}{16\pi G}\left(\frac{y_0}{2}\sqrt{\frac{2\tilde{c}_2 - 4ky_0}{3}} - \frac{y_0^2}{2l} + \frac{kly_0}{4} - \frac{l\tilde{c}_2}{24} \right) &(61) \\ 0 &= -\frac{y_0}{8\sqrt{3}\pi G}\sqrt{\frac{2\tilde{c}_2 - 4ky_0}{3}} - \frac{3ly_0^2}{4\pi G}\left(-\frac{6k}{\sqrt{3}y_0} + \frac{2\tilde{c}_2}{\sqrt{3}y_0^2} \right) + 6C\phi_0 &(62) \end{aligned}$$

and Eqs.(38) and (35) for the inner brane, when $k \neq 0$:

$$\begin{aligned} 8\tilde{b}' &= F_2(\tilde{y}_0) && (63) \\ 0 &= \frac{\tilde{y}_0}{8\sqrt{3}\pi G}\sqrt{\frac{2\tilde{c}_2 - 4k\tilde{y}_0}{3}} + \frac{3l\tilde{y}_0^2}{4\pi G}\left(-\frac{6k}{\sqrt{3}\tilde{y}_0} + \frac{2\tilde{c}_2}{\sqrt{3}\tilde{y}_0^2}\right) + 6\tilde{C}\tilde{\phi}_0 \ , && (64) \end{aligned}$$

Since p_2 is absorbed into the definition of $\tilde{c}_2$ in (61) and (63), Eqs.(62) and (64) can be regarded again as the equation which determines p_2 or $\phi_0 = p_1 \ln(p_2 y_0)$ ($\tilde{\phi}_0 = p_1 \ln(p_2\tilde{y}_0)$. Then

$$F_2'(y_0) = \frac{3}{16\pi G}\left(\frac{1}{2}\sqrt{\frac{2\tilde{c}_2 - 4ky_0}{3}} - \frac{\frac{ky_0}{3}}{\sqrt{\frac{2\tilde{c}_2 - 4ky_0}{3}}} - \frac{y_0}{l} + \frac{kl}{4}\right) \ . \qquad (65)$$

Then if $F_2'(y_0) = 0$, one gets

$$0 = f(y_0) \equiv \frac{4k}{l^2}y_0^3 + \left(k^2 - \frac{2\tilde{c}_2}{l^2}\right)y_0^2 + \left(\frac{k^3l^2}{4} - \tilde{c}_2 k\right)y_0 + \left(-\frac{k^2l^2}{8} + \frac{1}{3}\right)\tilde{c}_2 \qquad (66)$$

and

$$f'(y_0) = \frac{12k}{l^2}y_0^2 + 2\left(k^2 - \frac{2\tilde{c}_2}{l^2}\right)y_0 + \left(\frac{k^3l^2}{4} - \tilde{c}_2 k\right) \ . \qquad (67)$$

Then if we further put $f'(y_0) = 0$, the determinant D of the equation is given by

$$\begin{aligned} \frac{D}{4} &= \left(k^2 - \frac{2\tilde{c}_2}{l^2}\right)^2 - \frac{12k}{l^2}\left(\frac{k^3l^2}{4} - \tilde{c}_2 k\right) \\ &= \frac{4}{l^2}\left\{\tilde{c}_2 + k^2l^2\left(1 + \sqrt{\frac{3}{2}}\right)\right\}\left\{\tilde{c}_2 + k^2l^2\left(1 - \sqrt{\frac{3}{2}}\right)\right\} \ . && (68) \end{aligned}$$

If $D < 0$, the equation $f'(y_0) = 0$ does not have any solution. Then there can be only one solution $f(y_0) = 0$, then in this case, $F_2(y_0)$ can have only one extremum. The explicit solutions of (66) are given by

$$\begin{aligned} y_0 &= -\frac{l^2k(1 - 2\hat{c})}{12k} + \left(-\beta + \sqrt{\beta^2 + \alpha^3}\right)^{\frac{1}{3}}\omega + \left(-\beta - \sqrt{\beta^2 + \alpha^3}\right)^{\frac{1}{3}}\omega^2 \\ \omega &= 1,\ \mathrm{e}^{\frac{2\pi i}{3}},\ \mathrm{e}^{\frac{4\pi i}{3}} \end{aligned}$$

$$\begin{aligned}
\alpha &\equiv \frac{2 - 8\hat{c} - 4\hat{c}^2}{3} \\
\beta &\equiv \frac{-7 - 12\hat{c} + 96\hat{c}^2 - 16\hat{c}^3}{54} \\
\hat{c} &= \frac{\tilde{c}_2}{k^2 l^2} \ .
\end{aligned} \tag{69}$$

Then if

$$\beta^2 + \alpha^3 < 0 \ , \tag{70}$$

Eq.(66) has three different real solutions, and $F_2(y_0)$ can have three extrema (at maximum).

Let us consider the solution of Eq.(61) or the behavior of $F_2(y_0)$ in more detail.

In case of $k > 0$, since $F_2(y_0)$ contains $\sqrt{\frac{2\tilde{c}_2 - 4ky_0}{3}}$, the value of y_0 is restricted to be $0 \leq y_0 \leq \frac{\tilde{c}_2}{2k}$ and $\tilde{c}_2$ should be positive: $\tilde{c}_2 > 0$. Since

$$\begin{aligned}
F_2(0) &= -\frac{3}{16\pi G}\frac{l\tilde{c}_2}{24} < 0 \\
F_2\left(\frac{\tilde{c}_2}{2k}\right) &= -\frac{3}{16\pi G}\frac{1}{8k^2 l}\left(\tilde{c}_2 - \frac{2k^2 l}{3}\right)\tilde{c}_2 \ ,
\end{aligned} \tag{71}$$

Eq.(61) has a outer brane solution, at least, if $F_2\left(\frac{\tilde{c}_2}{2k}\right) \geq -8b'$. As usually $-8b' > 0$, $F_2\left(\frac{\tilde{c}_2}{2k}\right)$ should be positive, which requires $\tilde{c}_2 < \frac{2k^2 l}{3}$. More generally, since $F_2'(0) > 0$ and $F_2\left(y_0 \to \frac{\tilde{c}_2}{2k}\right) \to -\infty$, $F_2(y_0)$ can have at least one maximum. If the maximum is greater than $-8b'$ ($= 0$ in the classical case), there can be outer brane solution(s). And if $F_2(0) < 8\tilde{b}'$ ($= 0$ in the classical case), there can be inner brane solution(s). We should note that such inner and/or outer brane(s) soultion(s) can exist in general even if $b' = \tilde{b}' = 0$. Hence, the possibility of creation of inflationary two brane-world Universe occurs not only on quantum but also on classical level (depending on the choice of the parameters).

We now consider the case of $k < 0$ for (61) in the case 2. If $\tilde{c}_2 > 0$, y_0 can take a value from 0 to positive infinity: $0 \leq y_0 < \infty$. Since

$$\begin{aligned}
&F_2(0) = -\frac{3}{16\pi G}\frac{l\tilde{c}_2}{24} < 0 \ , \quad F_2'(0) = \frac{3}{16\pi G}\left(\sqrt{\frac{2\tilde{c}_2}{6}} + \frac{kl}{4}\right) \\
&F_2(y_0 \to +\infty) \to -\frac{3}{16\pi G}\frac{y_0^2}{2l} < 0 \ ,
\end{aligned} \tag{72}$$

if $\tilde{c}_2 > \frac{3k^2l^2}{8}$, $F_2(y_0)$ has at least one maximum. If the value of the maximum is larger than $-8b'$ ($=0$ in the classical case), there is always an outer brane solution. Even if $\tilde{c}_2 < \frac{3k^2l^2}{8}$, from (68), there can be a maximum when $\tilde{c}_2 > k^2l^2\left(\sqrt{\frac{3}{2}}-1\right)$. Here we should note $\frac{3}{8} > \sqrt{\frac{3}{2}}-1$. When $\tilde{c}_2 < k^2l^2\left(\sqrt{\frac{3}{2}}-1\right)$, $F_2(y_0)$ becomes a monotonically decreasing function of y_0. Since $F_2(0)$ is negative, there cannot be any outer brane solution.

In case that $k<0$ and $\tilde{c}_2<0$, we find $y_0 > \frac{\tilde{c}_2}{2k}$ in order that $f(y)$ is positive. The surface of $y_0 = \frac{\tilde{c}_2}{2k}$ can be regarded as a horizon. Since

$$\begin{aligned} F_2\left(\frac{\tilde{c}_2}{2k}\right) &= -\frac{3}{16\pi G}\frac{1}{8k^2l}\left(\tilde{c}_2 - \frac{2k^2l}{3}\right)\tilde{c}_2 < 0 \\ F_2\left(y_0 \to +\infty\right) &\to -\frac{3}{16\pi G}\frac{y_0^2}{2l} < 0 \ , \end{aligned} \tag{73}$$

and $F_2\left(y_0 \to \frac{\tilde{c}_2}{2k}\right) \to +\infty$, there is at least one maximum. If the maximum is larger then $-8b'$ (when $b'<0$), there are outer brane solutions. And if $F_2\left(\frac{\tilde{c}_2}{2k}\right) < 8b'$, there can be inner brane solution.

Finally, when $k=0$ in case 2 (32), the brane equation (37) has the following form:

$$0 = y_0^2 - ly_0\sqrt{\frac{2\tilde{c}_2}{3}} + \frac{2l^2\tilde{c}_2}{3} \ . \tag{74}$$

Then $\tilde{c}_2$ should be positive. The solution of (74) is given by

$$y_0 = \frac{l\sqrt{\tilde{c}_2}}{2}\left(\sqrt{\frac{2}{3}} \pm \frac{1}{\sqrt{3}}\right) \ . \tag{75}$$

Since both of the two solutions are positive, there can be a solution with both of inner and outer branes.

Case 3 solution.

We now briefly consider case 3 (33). First one should note that $p_2>0$ since the solution (33) contains $\sqrt{p_2}$. Then if we define $\tilde{c}_1$ by

$$\tilde{c}_1 \equiv \frac{c_1}{\sqrt{p_2}} \ , \tag{76}$$

when $k \neq 0$, Eqs.(36) and (34) have the following form:

$$-8b' = F_3(y_0)$$

$$
\begin{aligned}
&\equiv \quad \frac{3}{16\pi G}\left\{\frac{y_0}{2y_0}\sqrt{\frac{8\sqrt{y_0}\left(\tilde{c}_1 y_0 + 7k\sqrt{y_0}\right)}{21}}\right. \\
&\qquad \left. -\frac{y_0^2}{2l} - \frac{l}{24}\sqrt{y_0}\left(\tilde{c}_1 y_0 + 3k\sqrt{y_0}\right)\right\} \qquad (77) \\
0 &= \frac{y_0}{16\pi G}\sqrt{\frac{2\sqrt{y_0}\left(\tilde{c}_1 y_0 + 7k\sqrt{y_0}\right)}{7}} \\
&\qquad -\frac{l\sqrt{3y_0}}{4\pi G}\left(3kp_2\tilde{c}_1 y_0 + \frac{2c_2\sqrt{y_0}}{p_2}\right) - 3\sqrt{3}C\phi_0 \ . \qquad (78)
\end{aligned}
$$

Since p_2 is absorbed into the definition of $\tilde{c}_1$ in (77), Eq.(78) can be regarded as the equation which determines p_2 or $\phi_0 = p_1 \ln(p_2 y_0)$.

When $\tilde{c}_1$ is negative, we find $k > 0$ and $0 < y_0 < \frac{49k^2}{\tilde{c}_1^2}$ in order that $F_3(y_0)$ is real. Since

$$
\begin{aligned}
F_3\left(y_0 \to 0\right) &= -\frac{3}{16\pi G}\frac{lky_0}{8} < 0 \\
F_3\left(\frac{49k^2}{\tilde{c}_1^2}\right) &= \frac{3}{16\pi G}\frac{(7k)^3}{\tilde{c}_1^2}\left(\frac{l}{42} - \frac{7k}{2l\tilde{c}_1^2}\right) \ . \qquad (79)
\end{aligned}
$$

Then if $F_3\left(\frac{49k^2}{\tilde{c}_1^2}\right) > -8b'$ ($= 0$ in the classical case), there can be outer brane solution.

When $\tilde{c}_1$ is positive, y_0 can take a value from 0 to $+\infty$ if k is positive. Since

$$
F_3\left(y_0 \to 0\right) = -\frac{3}{16\pi G}\frac{lky_0}{8} < 0 \ , \quad F_3\left(y_0 \to +\infty\right) = -\frac{3}{16\pi G}\frac{y_0^2}{2l} < 0 \ , \qquad (80)
$$

it is not so clear if there can be any outer brane solution.

When $\tilde{c}_1 > 0$ and $k < 0$, we find $y_0 > \frac{49k^2}{\tilde{c}_1^2}$ in order that $F_3(y_0)$ is real. Since

$$
\begin{aligned}
F_3\left(\frac{49k^2}{\tilde{c}_1^2}\right) &= \frac{3}{16\pi G}\frac{(7k)^3}{\tilde{c}_1^2}\left(\frac{l}{42} - \frac{7k}{2l\tilde{c}_1^2}\right) > 0 \\
F_3\left(y_0 \to +\infty\right) &= -\frac{3}{16\pi G}\frac{y_0^2}{2l} < 0 \ , \qquad (81)
\end{aligned}
$$

there always exists outer brane solution if $F_3\left(\frac{49k^2}{\tilde{c}_1^2}\right) > -8b'$.

From the above results in case 1∼3, we find there very often appear two (inner and outer) branes solution as in the first model by Randall and Sundrum [1]. Moreover, the branes may be curved as de Sitter or hyperbolic space which gives the way for ever expanding inflationary Universe. Such solutions often can exist even if there is no any quantum effect, i.e., $b' = 0$.

Let us make few remarks on the form of metric. If one considers the metric in the form (1), the warp factor $\mathrm{e}^{2\tilde{A}(z)}$ does not behave as an exponential function of z but as a power of z. For example, in case 1 (31),

$$\begin{aligned} ds^2 &= \frac{dy^2}{q^2 y} + y \sum_{i,j=1}^{4} \hat{g}_{ij} dx^i dx^j \\ &= dz^2 + \frac{z^2}{4q} \sum_{i,j=1}^{4} \hat{g}_{ij} dx^i dx^j \end{aligned} \tag{82}$$

where $z = 2q\sqrt{y}$. In case 2 (32)

$$\begin{aligned} ds^2 &= \frac{3dy^2}{2\tilde{c} - 4ky} + y \sum_{i,j=1}^{4} \hat{g}_{ij} dx^i dx^j \\ &= dz^2 + \left(\frac{\tilde{c}_2}{2} - \frac{2k^2 z^2}{3k}\right) \sum_{i,j=1}^{4} \hat{g}_{ij} dx^i dx^j \ , \end{aligned} \tag{83}$$

where $z = \frac{1}{k}\sqrt{\frac{3\tilde{c}_2}{2} - 3ky}$. One should note that the exponential behavior of the warp factor $\mathrm{e}^{A(z)} \sim \mathrm{e}^{\gamma z}$ requires $f(y) \sim \frac{1}{y^2}$, which tells that the spacetime is nearly AdS:

$$\begin{aligned} ds^2 &\sim \frac{dy^2}{\gamma^2 y^2} + y \sum_{i,j=1}^{4} \hat{g}_{ij} dx^i dx^j \\ &= dz^2 + \mathrm{e}^{\gamma z} \sum_{i,j=1}^{4} \hat{g}_{ij} dx^i dx^j \ , \end{aligned} \tag{84}$$

where $y = \mathrm{e}^{\gamma z}$. This would require that we need a region (of complete spacetime) where, the potential and the dilaton become almost constant. It results in difficulties when one tries to explain the hierarchy using this model.

Hence, we presented number of dilatonic (inflationary, flat or hyperbolic) two brane-world Universes which are created by quantum effects of brane matter. Sometimes, such Universes may be realized due to specific choice of dilatonic potential even on classical level.

4 Multi-brane generalization

In some papers (for example in [18]), the solution with many branes was proposed. In such model, there are two AdS spaces with the different radii or different values of the cosmological constants. They are glued by a brane, whose tension is given by the difference of the inverse of the radii. In the solution, the value of $\frac{dA}{dz}$ in the metric of the form (1) jumps at the brane, which tells the value of $f(y)$ in the metric choice in (17) jumps on the brane since $\sqrt{f(y)} = \frac{dz}{dy} = \frac{1}{2y\frac{dA}{dz}}$. Imagine one includes the quantum effects on the brane. Then one can, in general, glue two AdS-like spaces with same values of the cosmological constant. Let us assume that there is a brane at $y = \hat{y}_0$ and there are two AdS-like spaces in $y > \hat{y}_0$ and $y < \hat{y}_0$ glued by the brane. One now denotes the quantity in the AdS-like space in $y > \hat{y}_0$ ($y < \hat{y}_0$) by the suffix $+$ ($-$). If we consider the case where the value of l is identical in two AdS-like space and the value of the dilaton is continuous at the brane, we need not the counter term corresponding to S_1 in (5) or $\tilde{S}_1$ in (14) since the action corresponding to S_1 cancells with $\tilde{S}_1$ (note that the relative sign between S_1 and $\tilde{S}_1$ is opposite). Then instead of (34) and (36) or (35) and (38), one obtains

$$0 = -\frac{\hat{y}_0^2}{8\pi G}\left(\frac{\partial_y \phi_+(\hat{y}_0)}{\sqrt{f_+(\hat{y}_0)}} - \frac{\partial_y \phi_-(\hat{y}_0)}{\sqrt{f_-(\hat{y}_0)}}\right) + 6\hat{C}\hat{\phi}_0(\hat{y}_0) \tag{85}$$

$$0 = \frac{3\hat{y}_0}{16\pi G}\left(\frac{1}{2\sqrt{f_+(\hat{y}_0)}} - \frac{1}{2\sqrt{f_-(\hat{y}_0)}}\right) + 8\hat{b}' \tag{86}$$

for $k \neq 0$ cases. Here we denote the quantities on the brane by $\hat{\ }$. For $k \neq 0$, we cannot put a brane without making the cosmological constants in the AdS-like spaces $y > \hat{y}_0$ and $y < \hat{y}_0$ different as in the case of [18] where no quantum corrections case has been considered.

As an example, we only limit to case 1 in (31)

$$f_\pm = \frac{1}{q_\pm^2 y} \ . \tag{87}$$

Then using (86), one finds

$$0 = \frac{3\sqrt{\hat{y}_0}}{16\pi G}\left(\frac{1}{2q_+} - \frac{1}{2q_-}\right) + 8\hat{b}' \ . \tag{88}$$

On the other hand, from (85), we obtain

$$0 = -\frac{\hat{y}_0^{\frac{3}{2}}}{8\pi G}\left(q_+ p_{1+} - q_- p_{1-}\right) + 6\hat{C}\phi(\hat{y}_0) \ . \tag{89}$$

The condition of the continuity of the dilaton field at the brane gives, from (29),

$$\phi(\hat{y}_0) = p_{1+}\ln(p_{2+}\hat{y}_0) = p_{1-}\ln(p_{2-}\hat{y}_0) \ . \tag{90}$$

The equations (88), (89) and (90) are compatible with each other (Note that $q_\pm$ is given in terms of $p_{1\pm}$ by (39). Let $\hat{y}_0$ and q_+ (or p_{1+}) be independent parameters. Then Eq.(88) can be solved with respect to q_-:

$$q_- = q_-\left(q_+, \hat{y}_0\right) \equiv \frac{1}{\frac{1}{q_+} + \frac{16\pi G\cdot 16\hat{b}'}{3\sqrt{\hat{y}_0}}} \ . \tag{91}$$

Then putting $\phi(\hat{y}_0) = p_{1+}\ln(p_{2+}\hat{y}_0)$ in (89), we can solve the equation with respect to p_{2+}

$$p_{2+} = p_{2+}\left(q_+, \hat{y}_0\right) \equiv \frac{1}{y_0}\mathrm{e}^{\frac{1}{6\hat{C}p_1(q_1)}\cdot\frac{\hat{y}_0^{\frac{3}{2}}}{8\pi G}\left(q_+ p_{1+}(q_+) - q_-(q_+,\hat{y}_0)p_{1-}(q_-(q_+,\hat{y}_0))\right)} \ . \tag{92}$$

Here $p_{1\pm}\left(q_\pm\right)$ is defined by solving (39):

$$p_{1\pm}\left(q_\pm\right) = \sqrt{\frac{3}{2} - \frac{2k}{q_\pm^2}} \ . \tag{93}$$

Finally (90) can be solved with respect to p_{2-}:

$$p_{2-} = \frac{\left(p_{2+}\left(q_+, \hat{y}_0\right)\hat{y}_0\right)^{\frac{p_1(q_1)}{p_{1-}(q_-(q_+,\hat{y}_0))}}}{y_0} \ . \tag{94}$$

When $k > 0$, Eq.(93) gives a constraint

$$q_\pm^2 > \frac{4k}{3} \ . \tag{95}$$

As long as the constraint in (95) holds, iterating the above procedure, we can obtain curved multi-brane solutions. Hence, we outlined the way to generalize two brane-world for multi-brane case.

5 Discussion

In summary, we presented the generalization of quantum dilatonic brane-world[10] where brane is flat, spherical (de Sitter) or hyperbolic and it is induced by quantum effects of CFT living on the brane. In this generalization one may have two brane-worlds or even multi-brane-worlds which proves general character of scenario suggested in refs.[5, 6] where instead of arbitrary brane tension added by hands the effective brane tension is produced by boundary quantum fields. What is more interesting the bulk solutions have analytical form, at least, for specific choice of bulk potential under consideration.

In classical dilatonic gravity the variety of brane-world solutions has been presented in ref.[16] where also the question of singularities has been discussed. The fine-tuned example of bulk potential where one gets bulk solution which is not singular has been presented. Let us consider if our solutions contain the curvature singularity or not. Multiplying $g_{(5)}^{\mu\nu}$ with the Einstein equation in the bulk:

$$R_{(5)\mu\nu} - \frac{1}{2}\nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{(5)\mu\nu}\left(R_{(5)} - \frac{1}{2}\nabla_\rho\phi\nabla^\rho\phi + \frac{12}{l^2} + \Phi(\phi)\right) , \tag{96}$$

which is obtained from $S_{\rm EH}$ in (2), one gets

$$R_{(5)} = \frac{1}{2}\nabla_\rho\phi\nabla^\rho\phi - \frac{5}{3}\left(\frac{12}{l^2} + \Phi(\phi)\right) . \tag{97}$$

Substituting expressions (29) and (30) into (97), we find

$$R_{(5)} = \frac{p_1^2}{2y^2 f} - \frac{5}{3}\left(c_1(p_2y)^{ap_1} + c_2(p_2y)^{2ap_1}\right) . \tag{98}$$

Then for cases $1 \sim 3$, the scalar curvature $R_{(5)}$ is given by

$$\text{case 1} \qquad R_{(5)} = -\frac{3}{2}\frac{p_1^2q^2}{y} \tag{99}$$

$$\text{case 2} \qquad R_{(5)} = \frac{8k}{y} - \frac{2\tilde{c}_2}{3y^2} \tag{100}$$

$$\text{case 3} \qquad R_{(5)} = -\frac{33\tilde{c}_1}{21\sqrt{y}} - \frac{4k}{y} . \tag{101}$$

In all cases the singularity appears at $y = 0$.

In case 1, when $y \sim 0$ and the coordinates besides y are fixed, the infinitesimally small distance ds is given by

$$ds = \sqrt{f} dy \sim \frac{dy}{q\sqrt{y}} \ , \tag{102}$$

which tells that the distance between the brane and the singularity is finite. Then in cases of $k = 0$ and $k < 0$, the singularity is naked when we Wick re-rotate spacetime to Lorentzian signature. When $k > 0$, the singularity is not exactly naked after the Wick re-rotation since the horizon is given by $y = 0$, i.e. the horizon coincides with the curvature singularity.

In case 2, the situation is not changed for $k = 0$, $k > 0$ and $k < 0$ with $\tilde{c}_2 > 0$ from that in case 1 and the distance between the brane and the singularity is finite since $ds \sim \frac{dy}{\sqrt{y}}\sqrt{\frac{3}{2\tilde{c}_2}}$ when y is small. When $k < 0$ with $\tilde{c}_2 < 0$, however, the singularity is not naked since there is a kind of horizon at $y = \frac{\tilde{c}_2}{2k}$, where $\frac{1}{f(y)} = 0$. We should note the scalar curvature $R_{(5)}$ in (100) is finite. This tells that y is not proper coordinate when $y \sim \frac{\tilde{c}_2}{2k}$. If new coordinate η is introduced

$$\eta^2 \equiv 2\left(y - \frac{\tilde{c}_2}{2k}\right) \ , \tag{103}$$

the metric in (17) is rewritten as follows,

$$ds^2 = -\frac{3}{4k} d\eta^2 + \left(\frac{\tilde{c}_2}{2k} + \frac{\eta^2}{2}\right)\sum_{i,j=1}^{4} \hat{g}_{ij}(x^k) dx^i dx^j \ . \tag{104}$$

The radius of 4d manifold with negative k, whose metric is given by $\hat{g}_{ij}$, has a minimum $\frac{\tilde{c}_2}{2k}$ at $\eta = 0$, which corresponds to $y = \frac{\tilde{c}_2}{2k}$. The radius increases when $|\eta|$ increases. Therefore the spacetime can be regarded as a kind of wormhole, where two universes corresponding to $\eta > 0$ and $\eta < 0$, respectively, are joined at $\eta = 0$.

In case 3, the singularity is naked (the singularity is not exactly naked when $k > 0$ as in case 1) in general and the distance between the brane and the horizon is finite except $k > 0$ and $\tilde{c}_1 < 0$ case since there is a horizon at $\sqrt{y} = -\frac{7k}{\tilde{c}_1}$ where the scalar curvature (101) is finite.

The price for having analytical bulk results (exactly solvable bulk potential) is the presence of (naked) singularity. One can, of course, present the fine-tuned examples of bulk potential as in refs.[10, 16] where the problem of singularity does not appear. Moreover, bulk quantum effects may significally modify classical bulk configurations [6, 19, 20] which presumbly may help in resolution of (naked) singularity problem. However, in such situation there are no analytical bulk solutions in dilatonic gravity.

There are various ways to extend the results of present work. First of all, one can construct multi-brane dilatonic solutions within the current scenario for another classes of bulk potential. However, this requires the application of numerical methods. Second, it would be interesting to describe the details of brane-world anomaly driven inflation (with non-trivial dilaton) at late times when it should decay to standard FRW cosmology. Third, within similar scenario one can consider dilatonic brane-world black holes which are currently under investigation.

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